AD-A101 436 PENNSYLVANIA UNIV PHILADELPHIA COLL OF ENGINEERING A--ETC F/6 20/11
EFFECTIVE ELASTIC PROPERTIES OF CRACKED MATERIALS.(U)
MAY 81 T GOTTESMAN, Z HASHIN, M A BRULL N00014-78-C-0544

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NAVAL AIR SYSTEMS

OFFICE OF NAVAL RESEARCH

CONTRACT N00014-78-C-0544

Technical ~~Report~~ No. 6

AD A101436

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by

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May 1981



81 7 15 016

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER 6 | 2. GOVT ACCESSION NO. AD-A101 436 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) EFFECTIVE ELASTIC PROPERTIES OF CRACKED MATERIALS | | 5. TYPE OF REPORT & PERIOD COVERED Technical |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) T. Gottesman, Z. Hashin, and M.A. Brull | | 8. CONTRACT OR GRANT NUMBER(s) N00014-78-C-0544, |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS University of Pennsylvania Philadelphia, PA 19104 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS |
| 11. CONTROLLING OFFICE NAME AND ADDRESS Naval Air Systems Command, Washington, D.C. Office of Naval Research, Arlington, VA | | 12. REPORT DATE May, 1981 |
| | | 13. NUMBER OF PAGES |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | | 15. SECURITY CLASS. (of this report) |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; Distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Cracks, Crack Density, Effective Elastic Moduli, Fatigue Damage, Fiber Composites, Lower Bounds, Orthotropic Materials, Stiffness Reduction, Stress Intensity Factor, Upper Bounds, Variational Principles

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

# EFFECTIVE ELASTIC PROPERTIES OF CRACKED MATERIALS

T. Gottesman(*), Z. Hashin(**) and M. A. Brull(***)

## ABSTRACT

The problem of the determination of effective elastic properties of materials containing many interacting cracks has been considered by variational methods. It has been shown that effective elastic moduli expressions for low crack density are general upper bounds for the case of arbitrary crack density. Lower bounds on effective moduli have been obtained on the basis of admissible, stress fields which are expressed in terms of known crack field solutions. Upper and lower bounds are reasonably close together for practically significant range of crack density.

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## 1. Introduction

The present work is concerned with the determination of the effective elastic properties of a body containing many interacting flat cracks. The problem is of significant practical importance. For example: it has been observed that crack density in rocks increases before the occurrence of an earthquake. If it were possible to determine the stiffness reduction due to cracks, analytically, the measurement of rock elastic moduli could provide information about the density of crack accumulation. Another example concerns fatigue damage. In unidirectional fiber composites fatigue damage frequently accumulates in the form of cracks parallel to to fibers. This reduces the elastic moduli of the composite, a process which has become known as "wearout". The crack density, thus a measure of the internal fatigue damage, could be easily determined by elastic moduli reduction measurement if a quantitive relation between moduli and crack density were available.

The problem under consideration has received repeated attention. Early work was primarily concerned with the case of low crack density which implies that cracks are sufficiently far removed from one another so that the stress field around each can be approximated by the stress field around an isolated crack in an infinite medium. The first work of this nature appears to be that of Bristow [1] who treated the plane case of long rectangular aligned cracks extending indefinitely on $x_3$ direction and randomly oriented in the $x_1$, $x_2$ plane. Walsh [2,3,4] solved the case of randomly oriented plane elliptical cracks. Salganic [5] reconsidered the same problem by a dislocation approach. Knopoff and Garbin [6,7] treated this problem in terms of scattering of long waves by the cracks. Piau [8] used such an approach for the problem of aligned circular cracks. It should be pointed out that the scattering approach is

very complicated and that the same results are obtained in much easier fashion by static methods.

The case of interacting cracks is very difficult. The only exact result available appears to be due to Delameter, Herrmann and Barnett [9] who treated the case of a periodic rectangular plane array of cracks in terms of periodic dislocation fields and solved the resulting integral equations numerically.

The self consistent scheme (SCS) approximation has been applied by Budiansky and O'Connell [10] to assess the effective elastic properties of an isotropic body containing randomly oriented elliptical cracks. The same method was used by Hoenig [11] who considered oriented circular cracks in isotropic material. Levin [12] gave an approximate treatment using the Green tensor, assuming that the influence of any crack on the other cracks can be replaced by the influence of a force dipole situated at the crack center.

In view of the great analytical difficulty involved in direct computation of effective moduli the problem is here considered primarily by variational methods which permit establishment of bounds on the effective moduli. Here we shall also be concerned with the case of orthotropic matrix which is of importance for fiber composites. All of the available treatments in the literature seem to be confined to isotropic matrix.

## 2. Formulation

An elastic body contains a large number M of volumeless flat stable cracks whose distribution is statistically homogeneous. The external surface S is subjected to the homogeneous boundary conditions

$$T_i(S) = \sigma^o_{ij} n_j = T^o_i \tag{2.1}$$

where $\sigma^o_{ij}$ are constant stresses. In the absence of cracks, $\sigma^o_{ij}$ is the stress field in the body. In the presence of cracks the stress field becomes very complicated since it must in addition satisfy the conditions

$$T_i(S_m) = 0 \tag{2.2}$$

on all crack surfaces $S_m$. However, by the average stress theorem, [13],

$$\bar{\sigma}_{ij} = \sigma^o_{ij} \tag{2.3}$$

where here and from now on overbar denotes average over the entire body, and by statistical homogeneity also over any representative volume element (RVE) containing sufficiently many cracks.

The effective elastic properties of a composite material or heterogeneous medium are defined by the relations

$$\bar{\sigma}_{ij} = C^*_{ijkl}\,\bar{\varepsilon}_{kl} \quad \text{(a)}$$
$$\bar{\varepsilon}_{ij} = S^*_{ijkl}\,\bar{\sigma}_{kl} \quad \text{(b)} \tag{2.4}$$

where $C^*_{ijkl}$ and $S^*_{ijkl}$ are effective moduli and compliances, respectively, $\bar{\sigma}_{ij}$ are given by (2.3) and $\bar{\varepsilon}_{ij}$ are the average strains. It should be noted that for a material containing holes or cracks $\bar{\varepsilon}_{ij}$ are average strains taken over matrix <u>and</u> holes as if the latter were deforming continua.

Decomposing average strains and stresses into weighted averages in terms of phase region averages and their volume fractions and using the elastic matrix stress-strain relations it follows that

$$S^*_{ijkl} = S^{(1)}_{ijkl}\,\bar{\sigma}_{kl} + \bar{\varepsilon}_{ij}^{(2)} v_2 \tag{2.5}$$

where 1 denotes matrix and 2 denotes holes of any shape. The average strain theorem (see e.g. [13]) can be used to transform $\bar{\varepsilon}_{ij}^{(2)}$ into a surface integral over the holes. In the special case when the holes become flat volumeless cracks (2.5) assumes the form

$$S^{*}_{ijkl}\,\bar{\sigma}_{kl} = S^{(1)}_{ijkl}\,\bar{\sigma}_{kl} + \gamma_{ij} \qquad \text{(a)}$$

$$\gamma_{ij} = \frac{1}{2V}\sum_{m}\int_{S_m}([u_i]n_j + [u_j]n_i)\,dS \qquad \text{(b)} \qquad (2.6)$$

where $S_m$ is the surface of the mth crack and $[u_i]$ are the displacement jumps across the crack surfaces.

If the body is statistically isotropic with isotropic matrix it follows from (2.6) that the effective bulk modulus $K^*$ and the effective shear modulus $G^*$ are given by

$$\frac{1}{K^*} = \frac{1}{K_1} + \frac{2}{\sigma^o}\,\gamma_{jj}$$
$$\frac{1}{G^*} = \frac{1}{G_1} + \frac{2}{\sigma_{12}^o}\,\gamma_{12} \qquad (2.7)$$

where $\sigma^o$ is an applied average isotropic stress and $\sigma^o_{12}$ is an applied average shear stress. Expressions of type (2.7) can easily be written down for cases of anisotropic matrix and statistical anisotropy dictated by crack distribution.

If follows that the effective elastic properties of cracked bodies are determined by the crack opening displacements. To determine these it is in general necessary to find the entire displacement field in the cracked body subject to (2.1) and (2.2). This, of course, is an enormously difficult problem.

Effective elastic moduli of heterogeneous materials can alternatively and equivalently be defined in terms of elastic energy. The stress energy stored in any heterogeneous body subject to (2.1) is rigorously given by, [13].

$$U = \frac{1}{2} S^{*}_{ijkl} \sigma^{o}_{ij} \sigma^{o}_{kl} V = U_o + \sum_m \delta U_m \tag{2.8}$$

where

$$U_o = \frac{1}{2} S_{ijkl} \sigma^{o}_{ij} \sigma^{o}_{kl} V \tag{2.9}$$

and $\delta U_m$ is the energy change due to the $m^{th}$ crack in the presence of all the others.

According to a well known result, Eshelby, [14], $\delta U_m$ for a hole can be written as

$$\delta U_m = \frac{1}{2} \int_{S_m} \sigma^{o}_{ij} u_i n_j dS \tag{2,10}$$

which in the case of a crack becomes

$$\delta U_m = \frac{1}{2} \sigma^{o}_{ij} \int_{S_m} [u_i] n_j dS \tag{2.11}$$

Eqns. (2.8-11) may be summarized in the form

$$U = U_o + \frac{1}{2}\sigma^o_{ij}\gamma_{ij}V \tag{2.12}$$

which is the equivalent of (2.6).

Alternatively (2.11) can be expressed in terms of stress intensity factor (SIF) integrals. For line cracks in plane isotropic elasticity

$$\delta U_m = \phi\ \frac{2\pi}{E}\int_0^{a_m}[K_I^2(x) + K_{II}^2(x) + (1-\nu)K_{III}^2(x)]dx \tag{2.13}$$

where $\phi = 1$ for plane stress, $\phi = 1-\nu^2$ for plane strain, $a_m$ is half the crack length and $K_I$, $K_{II}$, $K_{III}$ are SIF in modes I, II and III respectively. For an isolated crack in an infinite body (2.13) assumes the form

$$\delta U_m = \phi\ \frac{\pi}{E}\ a_m[K_I^2 + K_{II}^2 + (1-\nu)K_{III}^2] \tag{2.14}$$

Corresponding results are available for plane cracks of arbitrary plane shapes and in particular for elliptical cracks, [10].

## 3. Direct Approaches

The case which is most easily solved is low crack density. In this event it is assumed that cracks are sufficiently far apart so that the energy change due to any one crack is determined as if it were isolated in infinite matrix with the applied homogeneous stress field at infinity.

Examination of the form of $\delta U_m$ for plane and elliptical cracks reveals that for low crack density (2.8-9) are equivalent to

$$[S^*_{ijkl} - (S_{ijkl} + \alpha\Gamma_{ijkl})]\sigma^o_{ij}\sigma^o_{kl} = 0 \tag{3.1}$$

where $\Gamma_{ijkl}$ depends on matrix properties and crack geometry and the crack density parameter is given by

$$\alpha = \frac{1}{A} \sum_m a_m^2 \qquad \text{line cracks} \qquad \text{(a)}$$

$$\alpha = \frac{1}{V} \sum_m a_m b_m^2 \qquad \text{elliptical cracks} \qquad \text{(b)} \qquad (3.2)$$

where in the first case $a_m$ is half the crack length and A is the area of the plane body while in the second case $a_m$, $b_m$ are major and minor axes of the elliptical crack respectively, while V is the volume. Since $\sigma^o_{ij}$ and $\sigma^o_{kl}$ are arbitrary stresses it follows that

$$\hat{S}^*_{ijkl} = S_{ijkl} + \alpha \Gamma_{ijkl} \qquad (3.3)$$

where ^ denotes, here and from now on, low crack density result.

As an example we consider a plane orthotropic sheet containing cracks aligned in $x_1$ direction, in a state of plane average stress $\sigma^o_{11}$, $\sigma^o_{22}$, $\sigma^o_{12}$. The strain-stress relations of the uncracked material are

$$\begin{aligned} \varepsilon_{11} &= \sigma_{11}/E_1 - \sigma_{22}\,\nu_{12}/E_1 \\ \varepsilon_{22} &= -\sigma_{11}\,\nu_{12}/E_1 + \sigma_{22}/E_2 \\ \varepsilon_{12} &= \sigma_{12}/2G_{12} \end{aligned} \qquad (3.4)$$

This may serve as a model for a unidirectionally fiber reinforced lamina in which case $E_1 = E_A$ - axial Young's modulus; $E_2 = E_T$ - transverse Young's modulus; $\nu_{12} = \nu_A$ - axial Poisson's ratio; $G_{12} = G_A$ - axial shear modulus.

The change in stress energy due to a crack of length $2a_m$ in an orthotropic sheet is given by, [15, 16]

$$\delta U_m = 2\pi \int_0^{a_m} [R_{22}K_I^2(x) + R_{12}K_{II}^2(x)]dx \tag{3.5}$$

where

$$R_{22}^2 = \frac{1}{2E_1^{1/2}E_2^{3/2}} + \frac{1}{4E_2}(1/G_{12} - 2\nu_{12}/E_1)$$

$$R_{12}^2 = \frac{1}{2E_1^{3/2}E_2^{1/2}} + \frac{1}{4E_1}(1/G_{12} - 2\nu_{12}/E_1) \tag{3.6}$$

For an isolated crack in an infinited sheet this becomes

$$\delta U_m = \pi a_m^2 (R_{22}\sigma^{o2}_{22} + R_{12}\sigma^{o2}_{12}) \tag{3.7}$$

Since the crack does not perturb a homogeneous $\sigma_{11}$ field this stress does not enter into the energy change. Comparison of (3.7) with (3.1) and (3.3), taking into account the orthotropy of cracked and uncracked material yields

$$\frac{E_2^*}{E_2} = \frac{1}{1 + \pi\{2E_2[(E_1 E_2)^{-1/2} + 1/2G_{12} - \nu_{12}/E_1]\}^{1/2}\alpha} \quad \text{(a)}$$

$$\frac{G_{12}^*}{G_{12}} = \frac{1}{1 + \pi G_{12}\{2/E_1[(E_1 E_2)^{-1/2} + 1/2G_{12} - \nu_{12}/E_1]\}^{1/2}\alpha} \quad \text{(b)} \tag{3.8}$$

$$E_1^* = E_1 \qquad \text{(a)}$$
$$\nu_{12}^* = \nu_{12} \qquad \text{(b)} \qquad (3.9)$$

where $\alpha$ is given by (3.2a).

It should be noted that (3.9) are exact results for any aligned crack distributions since such cracks do not perturb a homogeneous stress.

When the crack density is not low and thus crack interaction cannot be neglected the problem becomes extremely difficult. In view of the formulation given above it is necessary to either know the crack face displacements or their stress intensity factors. To obtain such information it is in general necessary to determine the displacement fields or the singular stress fields in the cracked body. The only exact treatment available seems to be due to Delameter, Herrmann and Barnett [9] for the case of a rectangular periodic array of plane cracks in isotropic matrix. Using dislocation theory the problem was formulated in terms of integral equations which were solved numerically. ([9b] corrects numerical mistakes appearing in [9a]).

An approximate method known as the self consistent scheme (SCS) has first been applied to the problem by Budiansky and O'Connell [10]. The basic idea is very simple: it is assumed that the energy change associated with any crack can be estimated as if this crack were located in a homogeneous elastic body whose elastic properties are the effective elastic properties to be determined. The analytical consequence of such an assumption is as follows: if the dependence of $\Gamma_{ijkl}$ in (3.3) on matrix elastic moduli $C_{ijkl}$ or compliances $S_{ijkl}$ is denoted $\Gamma_{ijkl}(\underline{C})$ or

$\Gamma_{ijkl}(\underline{S})$ then

$$S^*_{ijkl} = S_{ijkl} + \alpha\Gamma_{ijkl}(\underline{S}^*) \tag{3.10}$$

where $S^*_{ijkl}$ are the effective compliances to be determined. Equs. (3.10) are a set of simultaneous equations to determine the effective compliances.

While this approach is simple and straightforward it must be realized that it is based on the drastic assumption that the crack tip "sees" the effective medium. This is of course incorrect since the crack tip sees matrix and other cracks. The concept of the effective medium is valid only for averages over RVE containing many cracks. It can certainly not be applied to the immediate vicinity of a crack tip where the stress gradients are extremely large. The energy change due a a crack may be accurately estimated by the SCS method when the crack is by an order of magnitude larger than all others, but not necessarily when all are of same order of magnitude.

As an example for (3.10) we consider again the cracked orthotropic sheet with cracks aligned in $x_1$ direction. Combination of (3.10) with (3.8) gives the equations

$$\begin{aligned} \frac{E_2^*}{E_2} &= 1-\pi\{2E_2^*[(E_1E_2^*)^{-1/2} + 1/2G_{12}^* - \nu_{12}/E_1]\}^{1/2}\alpha \\ \frac{G_{12}^*}{G_{12}} &= 1-\pi G_{12}^*\{2/E_1[(E_1E_2^*)^{-1/2} + 1/2G_{12}^* - \nu_{12}^*/E_1]\}^{1/2}\alpha \end{aligned} \tag{3.11}$$

Eqs. (3.11) are equivalent to $8^{th}$ order algebraic equations for each of the effective moduli and therefore have multiple real roots. Solution of the equations for matrix properties typical of unidirectional glass epoxy fiber composite gave two different $E_2^*(\alpha)$ both mononotonically

decreasing with $\alpha$ and two different $G_{12}^*(\alpha)$ one of which decreases with $\alpha$ while the other increases with $\alpha$. The second $G_{12}^*(\alpha)$ is therefore physically unacceptable. To make a choice between the two $E_2^*(\alpha)$ it would be necessary to argue that the one associated with unacceptable $G_{12}^*(\alpha)$ via (3.11) is also unacceptable. Whether or not such an argument is convincing remains an open question.

It is of interest to note that similar problems arise with the SCS solution for aligned circular cracks given by Hoenig [11], but only a single numerical solution has been presented in his paper.

Finally it is noted that the SCS may easily be generalized to the case when there are aligned cracks in $x_1$ and $x_2$ directions. The results which have been given in [17] are:

$$2(E_1\eta_1/E_2\eta_2)^{1/2} + \eta_1(E_1/G_{12}\gamma - 2\nu_{12}) - [(1-\eta_1)\pi\alpha_2]^{1/2} = 0$$

$$2(E_2\eta_2/E_1\eta_1)^{1/2} + \eta_2(E_2/G_{12}\gamma - 2E_2\nu_{12}/E_1) - [(1-\eta_2)/\pi\alpha_1]^{1/2} = 0 \qquad (3.12)$$

$$1/\gamma = 1 + \alpha_1 G_{12}/\alpha_2 E_1(1/\eta_1 - 1) + \alpha_2 G_{12}/\alpha_1 E_2(1/\eta_2 - 1)$$

where

$$\eta_1 = E_1^*/E_1 \qquad \eta_2 = E_2^*/E_2 \qquad \gamma = G_{12}^*/G_{12}$$

$$\alpha_1 = \sum_m a_{1m}^2/A \qquad \alpha_2 = \sum_m a_{2m}^2/A$$

and $a_{1m}$ and $a_{2m}$ are half crack lengths of cracks in $x_1$ and $x_2$ directions respectively. These equations also have multiple real roots.

## 4. Bounds for Effective Elastic Moduli

The problem of bounding the effective elastic properties of cracked materials will here be considered in terms of the classical extremum principles of minimum potential and complementary energies.

### Upper Bounds

To construct upper bounds we shall use the principle of minimum potential energy. The potential energy and the potential energy functional are defined by

$$U_P = \frac{1}{2} \int_V W \, dV - \int_{S_T} T_i u_i \, dS \qquad \text{(a)}$$

$$\tilde{U}_P = \frac{1}{2} \int_V \tilde{W} \, dV - \int_{S_T} T_i \tilde{u}_i \, dS \qquad \text{(b)} \qquad (4.1)$$

where W is the strain energy density, $T_i$ are prescribed tractions on $S_T$, $\tilde{u}_i$ is an admissible displacement field which must be continuous in the region excluding the cracks and satisfy displacement boundary conditions, if imposed, and

$$\tilde{\varepsilon}_{ij} = \frac{1}{2} (\tilde{u}_{i,j} + \tilde{u}_{j,i})$$

$$\tilde{\sigma}_{ij} = C_{ijkl} \tilde{\varepsilon}_{kl} \qquad (4.2)$$

$$\tilde{W} = \frac{1}{2} \tilde{\sigma}_{ij} \tilde{\varepsilon}_{ij}$$

The extremum principle states that

$$\tilde{U}_p \geq U_p \tag{4.3}$$

In the case of a cracked body it follows from (2.2) that the crack surfaces do not contribute to the surface integral in (4.1). Thus $S_T$ becomes the external surface S on which (2.1) is prescribed. Since $\sigma^o_{ij}$ are constant it follows at once that

$$\tilde{U}_p = \frac{1}{2}\int_V (\tilde{\sigma}_{ij} - 2\sigma^o_{ij})\tilde{\varepsilon}_{ij} dV \quad \text{(a)}$$

$$U_p = \frac{1}{2}\int_V (\sigma_{ij} - 2\sigma^o_{ij})\varepsilon_{ij} dV = -\frac{1}{2} S^*_{ijkl}\sigma^o_{ij}\sigma^o_{kl}V \quad \text{(b)} \tag{4.4}$$

The last equation (4.4b) following from the average theorem of virtual work, [13], and (2.4b). It is noted in passing that (4.4) apply for any heterogeneous body with (2.1) prescribed.

An admissible displacement field for a body containing an arbitrary distribution of cracks is here chosen in the form

$$\tilde{u}_i = u^o_i + \sum_m u'_{im} \tag{4.5}$$

where

$$u^o_i = \varepsilon^o_{ij} x_j \qquad \varepsilon^o_{ij} = S_{ijkl}\,\sigma^o_{kl} \tag{4.6}$$

is the displacement field in the absence of cracks and $u'_{im}$ is the perturbation field of the $m^{th}$ crack, as if all the other cracks were absent. Thus for cracks removed from the boundary $u^o_i+u'_{im}$ is the displacement

field when the crack is isolated in an infinite medium. The field (4.5) is the actual solution for low crack density. It is continuous in the region excluding the cracks and is thus an acceptable admissible field for that region. Across each crack (4.5) is discontinuous with jump $[u_1]$ of the displacement of the isolated crack. It should be noted that an admissible displacement field with no discontinuity across cracks would lead to the trivial result that the upper bounds for effective moduli are the matrix moduli.

The strains and stresses associated with (4.5) are

$$\begin{aligned} \tilde{\varepsilon}_{ij} &= \varepsilon^{o}_{ij} + \sum_m \varepsilon'_{ijm} \\ \tilde{\sigma}_{ij} &= \sigma^{o}_{ij} + \sum_m \sigma'_{ijm} \end{aligned} \qquad (4.7)$$

Since $\sigma'_{ijm}$ is the actual stress perturbation due to an isolated crack with (2.1) prescribed, each of the associated tractions vanish on the external boundary. Thus

$$T'_{im} = \sigma'_{ijm} n_j = 0 \qquad \text{on } S \qquad (4.8)$$

Introduction of (4.7) into (4.1b) yields the result

$$\tilde{U}_p = -U_o + \frac{1}{2}\int_V (\varepsilon^{o}_{ij} \sum_m \sigma'_{ijm} - \sigma^{o}_{ij} \sum_m \varepsilon'_{ijm} + \sum_m \sigma'_{ijm} \varepsilon'_{ijm})\, dV + \qquad \text{(a)}$$

$$+ \sum_m \sum_{\substack{n \\ m\neq n}} \int_V \sigma'_{ijm} \varepsilon'_{ijn}\, dV \qquad (4.9)$$

$$U_o = \frac{1}{2}\int_V \sigma^{o}_{ij} \varepsilon^{o}_{ij}\, dV \qquad \text{(b)}$$

It is easily shown that the $m^{th}$ term in the first integral of (4.9a) is $-\delta U_m$, the negative of the stress energy change due to an $m^{th}$ crack isolated in infinite medium. Therefore (4.9) assumes the form

$$\tilde{U}_p = -U_o - \sum_m \delta U_m + \sum_{\substack{m \\ m\neq n}} \sum_n \int_V \sigma'_{ijm} \varepsilon'_{ijn} \, dV \tag{4.10}$$

The stress $\sigma^o_{ij} + \sigma'_{ijm}$ is an actual stress field and therefore satisfies equilibrium. Since $\sigma_{ij}{}^o$ is constant, $\sigma'_{ijm}$ also satisfies equilibrium. *Similarly* $\varepsilon'_{ijn}$ *is a compatible strain field.* Therefore by virtual work

$$\int_V \sigma'_{ijm} \varepsilon'_{ijn} dV = \int_S T'_{in} u'_{in} dS + \int_{\text{on all cracks}} T'_{in} u'_{in} dS \tag{4.11}$$

The first integral on the right hand side vanishes because of (4.8). The second integral must be evaluated on the two adjoining faces of all cracks on which

$$T_i = \sigma_{ij} n_j + T'_{im} = 0$$

Thus the integrand becomes $-\sigma^o_{ij} u'_{in} n_j$ and since $u'_{in}$ is continuous across all cracks and since the signs of the normal are opposite on adjoining crack faces, the last integral in (4.11) also vanishes and thus (4.11) is zero. It then follows from (4.10) that

$$\tilde{U}_p = -U_o - \sum_m \delta U_m \tag{4.12}$$

Now $\delta U_m$ is the energy change due to <u>one isolated crack</u> in a large body. Most cracks are sufficiently removed from the boundary so that $\delta U_m$ can be computed as if the crack were at infinite distance from the boundary, thus by expressions of type (2.14). The cracks which are near the boundary constitute the usual boundary layer of a heterogeneous body. Their number relative to that of cracks removed from the boundary can be made indefinitely small by increase of the size of the cracked body. Therefore (4.12) can be determined to any desired accuracy by interpreting all $\delta U_m$ as energy changes of isolated cracks removed from the boundary. It should be noted that the same reasoning tacitly enters into derivation of all previous results such as low crack density, periodic crack arrays and SCS.

If follows from the low crack density analysis leading to (3.1), (3.3) that (4.12) can be written as

$$\tilde{U}_p = -\frac{1}{2} \hat{S}^*_{ijkl}\sigma^o_{ij}\sigma^o_{kl} \tag{4.13}$$

Then from (4.3) and the last of (4.15)

$$(S^*_{ijkl} - \hat{S}^*_{ijkl})\sigma^o_{ij}\sigma^o_{kl} \geq 0 \tag{4.14}$$

This positive definite form defines bounds for $S^*_{ijkl}$ in terms of the low crack density expressions $\hat{S}^*_{ijkl}$ as given by (3.3) in which, however, $\alpha$ is no longer small but arbitrary. Specializing (4.14) to the case of one normal zero stress, $\sigma^o_{22}$ say, it follows that $S^*_{2222}$ is smaller

than $\hat{S}^*_{2222}$. Since the reciprocals of these compliances are Young's moduli in 2 direction we have

$$E^*_2 \leq \hat{E}^*_2 = E_2^{*(+)} \tag{4.15}$$

Analogously, for one nonzero shear stress, $\sigma^o_{12}$ say, it follows that

$$G^*_{12} \leq \hat{G}^*_{12} = G_{12}^{*(+)} \tag{4.16}$$

Similar results are valid for the other Young's and shear moduli for *any* cracked material. Thus *the effective moduli expressions valid for low crack density are upper bounds for the effective moduli at any crack density*.

If the cracked material is statistically isotropic (4.14-16) become bounds for that case. It is also easily shown that the bulk modulus $K^*$ is bounded by

$$K^* \leq \hat{K}^* = K^{*(+)} \tag{4.17}$$

For randomly oriented circular cracks the results are,

$$\begin{aligned} E^{*(+)} &= \hat{E}^* = E/[1+16\alpha(1-\nu^2)(10-3\nu)/45(2-\nu)] \\ G^{*(+)} &= \hat{G}^* = G/[1+32\alpha(1-\nu)(5-\nu)/45(2-\nu)] \end{aligned} \tag{4.18}$$

where

$$\alpha = \frac{1}{V}\sum_m a_m{}^3$$

and $a_m$ is the radius of the $m^{th}$ crack. Thus (4.18) are upper bounds for $E^*$ and $G^*$ of a body containing an arbitrary number of circular cracks.

For an orthotropic sheet which contains an arbitrary number of cracks aligned in $x_1$ direction it follows from (4.15-16) that (3.8) become upper bounds for the moduli $E_2^*$ and $G_{12}^*$.

## Lower Bounds

To construct lower bounds we shall use the principle of minimum complementary energy. Since the imposed boundary conditions (2.1), (2.2) are traction boundary conditions the principle simplifies to that of minimum stress energy. The stress energy and the stress energy functional are defined by

$$U^{\sigma} = \frac{1}{2} \int_V S_{ijkl}\, \sigma_{ij}\, \sigma_{kl}\, dV = \frac{1}{2} S^*_{ijkl}\, \sigma^0_{ij}\, \sigma^0_{kl}\, V \qquad \text{(a)}$$

$$\tilde{U}^{\sigma} = \frac{1}{2} \int_V S_{ijkl}\, \tilde{\sigma}_{ij}\, \tilde{\sigma}_{kl}\, dV \qquad \text{(b)} \qquad (4.19)$$

where $\tilde{\sigma}_{ij}$ is an admissible stress field which must satisfy equilibrium and all traction boundary conditions.

The principle of minimum stress energy states

$$\tilde{U}^{\sigma} \geq U^{\sigma} \qquad (4.20)$$

The major difficulty in construction of $\tilde{\sigma}_{ij}$ is satisfaction of the crack boundary conditions (2.2). Consider a body with an arbitrary distribution of cracks, fig. 1. The body is subdivided into regions each of which contains one crack. Let it be assumed that the $m^{th}$ such region is subjected on its boundary $S_m$ to (2.1). The elastic stress field in this body satisfying (2.1) and (2.2) is denoted $\sigma^m_{ij}$. Then

evidently the stress field

$$\tilde{\sigma}_{ij} = \sigma^{m}_{ij} \qquad \text{in } V_m \tag{4,21}$$

is admissible for the entire body since it satisfies equilibrium and all boundary conditions.

It should be noted that the proposed $\sigma_{ij}$ is singular at the crack tips and the usual admissible fields considered in standard elasticity texts do not seem to include such cases. It may be argued that cracks are limiting cases of very flat holes with finite tip curvatures and no stress singularities. Then the stress energy functional (4.19b) must first be evaluated for stresses $\sigma^{m}_{ij}$ in regions containing such holes and the limit of these integrals are taken when the holes become cracks. These integrals are the actual strain energies stored in these regions subjected to (2.1) and each containing one crack and it is known that these energies remain finite. We can write for the $m^{th}$ region

$$\tilde{U}^{\sigma}_{m} = U^{\sigma}_{om} + \delta\tilde{U}^{\sigma}_{m}$$

where (4,22)

$$U^{\sigma}_{om} = 1/2\ S_{ijkl}\ \sigma^{o}_{ij}\ \sigma^{o}_{kl}\ V_m$$

and $\delta U_m$ can be expressed in terms of the stress intensity factor of the crack in the $m^{th}$ region, if known.

For line cracks and plane elasticity $\delta U_m$ will have the form (2.13). It is emphasized that the SIF entering into the expressions depend

on the shape of the $m^{th}$ region and its crack geometry.

If follows that

$$\hat{U} = U_o + \sum_m \delta\hat{U}_m$$

$$U_o = 1/2\, S_{ijkl}\, \sigma^o_{ij}\, \sigma^o_{kl}\, V \qquad (4.23)$$

A crack solution useful for present purposes is due to Isida [18] who considered the plane case of a finite rectangle with a central crack in Mode I, fig. 2. His expression for the SIF may be written

$$K_1 = \sigma_o a^{1/2} f_1(a/b, a/c) \qquad (4.24)$$

where $\sigma_o a^{1/2}$ is the mode I SIF for the same crack in plane stress when it is isolated in an infinite sheet. Numerical values of the function $f_1$ have been given in [18].

The same problem for an orthotropic rectangle has been treated numerically by Bowie and Freese [19]. The SIF for this case may be expressed as

$$K_1 = \sigma_o a^{1/2} f_2(a/b, a/c, \text{material properties}) \qquad (4.25)$$

Consider now a large plane specimen containing m randomly distributed cracks, all in $x_1$ direction, fig. 3. We can subdivide this specimen into rectangles, each rectangle containing a central crack or no crack as shown by the dashed lines. The aggregate of the solutions

of finite rectangles with dimensions $2b_m$ and $2c_m$ and central cracks of length $2a_m$ ($a_m$ can also vanish) under simple tension $\sigma_o$ is an admissible stress fields. In the special case when the ratios $a_m/b_m$, $a_m/c_m$ and $b_m/c_m$ remain constant throughout the specimen, eqs. (4.21 - 24) together with (2.13) give a lower bound on $E_2^*$ for isotropic matrix with aligned cracks in the form

$$\frac{E_2^*}{E} \geq [1 + \frac{4\pi\alpha}{a^2} \int_0^a f_1^2(a/b, a, c)\, a\, da]^{-1} \tag{4.26}$$

For orthotropic matrix we use (4.25) together with (3.5) and obtain a lower bound on $E^*_2$ in the form

$$\frac{E_2^*}{E} \geq \{ 1 + \frac{2^{3/2}\pi\alpha E_2^{1/2}}{a^2} [(E_1E_2)^{-1/2} + 1/2G_{12} - \nu_{12}/E_1]^{1/2} \cdot \int_0^a f_2^2(a/b,\ a/c,\ \text{material properties})\ a\ da \}^{-1} \tag{4.27}$$

The bounds (4.26) and (4.27) are evidently valid also for the case of a doubly periodic array of cracks, since in this case $a_m$, $b_m$ and $c_m$ are constant.

Another useful crack solution is due to Wilson [20], who considered the plane case of a finite isotropic rectangle with a central inclined crack under various loadings.

The solution for a finite rectangle, in tension or in shear, containing an inclined crack can be used to obtain lower bounds on $E^*$ and $G^*$ for a plane sheet with randomly distributed cracks. The solution for a finite rectangle with a central oblique crack subjected to uniform shear tractions on crack edges can be used to obtain a lower bound on $G_{12}^*$ for a plane isotropic sheet with aligned cracks.

Some of the results obtained are plotted in fig. 4 as a function of the crack density parameter $\alpha$. This parameter is the only geometrical information entering into the upper bounds (small crack density) and into the SCS. However, $\alpha$ is insufficient information to characterize the exact results given in [9] or the lower bounds. In both cases the results are based on specific geometries which are characterized by the ratios c/b and a/b. Then $\alpha = a^2/4bc$ and is clearly insufficient to specify the geometry. The c/b ratios used for lower bound calculations are indicated in the figures.

The upper and lower bounds are quite close for a respectable range of $\alpha$, up to $\alpha \simeq 0.15$. For small $\alpha(\alpha<0.05)$ all the results coincide. The SCS curves are close to the lower bounds. As the cracks become closer, i.e. for large a/b and small c/b, the distance between the bounds increases, and the small concentration result deviates further. This is reasonable, as the influence of the interactions between the cracks should become more important. The results are presented for a range of up to $\alpha = 0.3$, however the cracks may become unstable before $\alpha$ attains this value.

It has been found that in certain cases the results of [9] are above the general upper bound, which is unacceptable. The reason for this phenomenon is not known to us at the present time.

5.

The problem of analytical determination of elastic properties of cracked materials has been considered by variational methods. The motivation for this approach is that exact direct analysis of the problem is extremely difficult. The only exact solution available for arbitrary crack density is for a periodic crack array and had to be performed numerically.

The bounds obtained are reasonable close for crack densities of practical significance. When the bounds are close the small crack density results can be regarded as a good approximation since it has been shown that they are general upper bounds.

On the basis of the results obtained it appears that the SCS approximation tends to underestimate the effective elastic properties. A more serious problem with this method is that it can give non-unique results for effective elastic moduli.

The bounds derived in this work are based on known crack field solutions and require only small computational effort. It is to be expected that additional and perhaps better bounds can be derived as more crack solutions become available.

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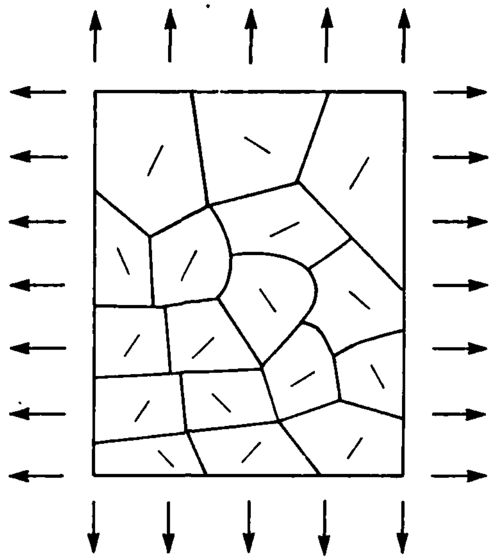

Fig. 1 - Subdivision of cracked body for construction of admissible stress fields: General case.

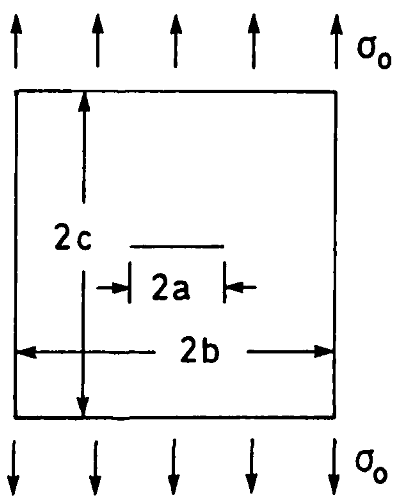


Fig. 2 - Finite rectangle with a central crack.

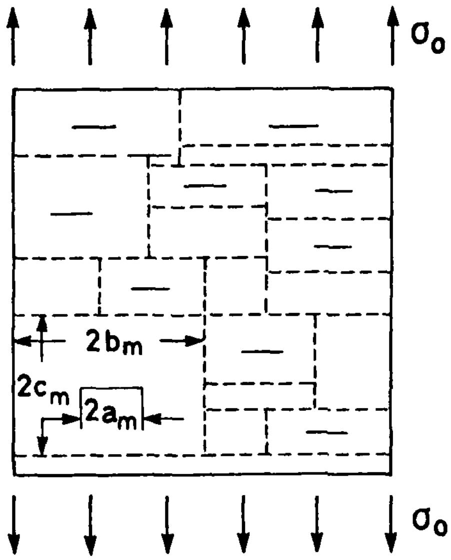


Fig. 3 - Subdivision of cracked body for construction of admissible fields: aligned plane cracks in a plane sheet.

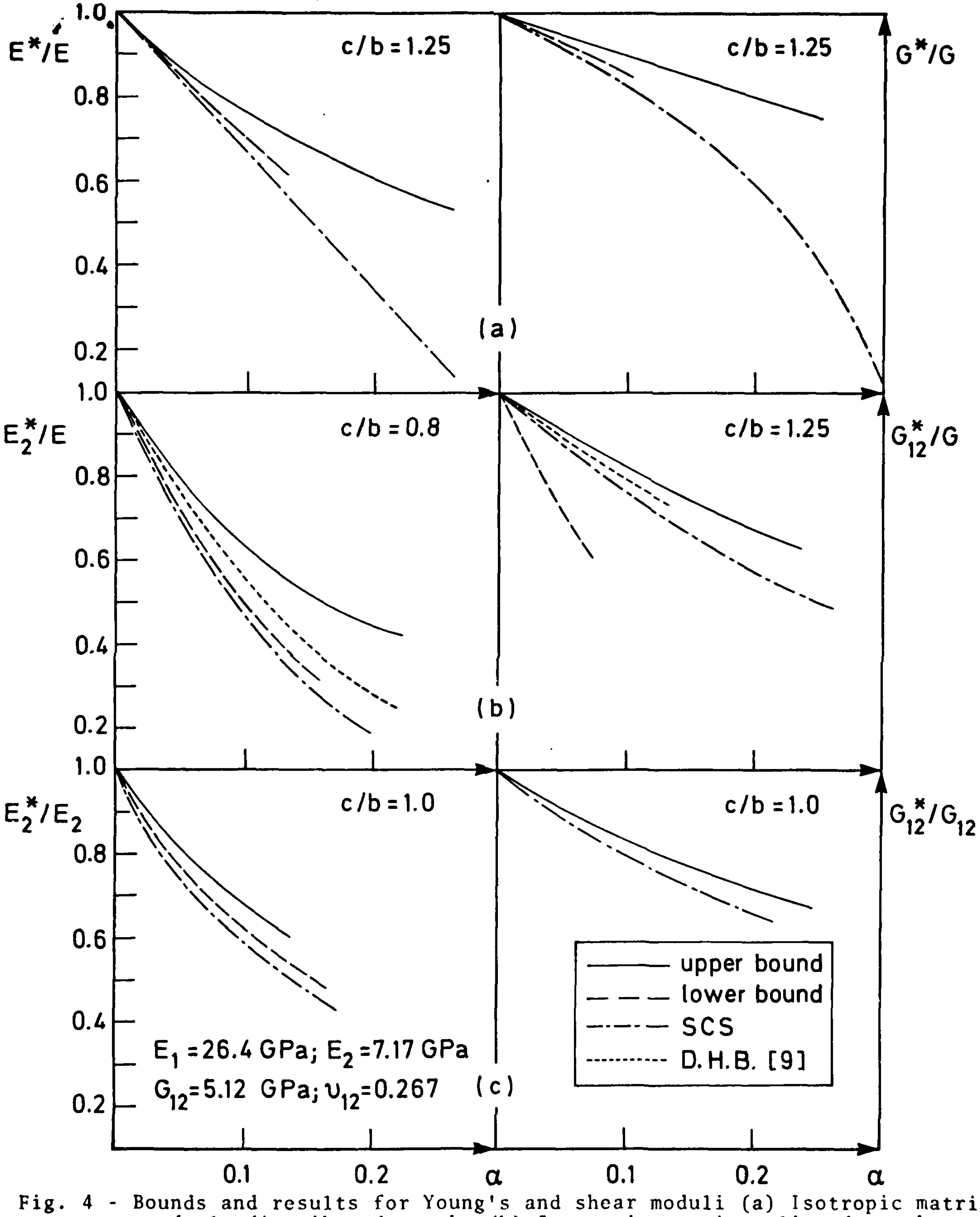


Fig. 4 - Bounds and results for Young's and shear moduli (a) Isotropic matrix, randomly distributed cracks (b) Isotropic matrix, aligned cracks (c) Orthotropic matrix (glass/epoxy composite $E_1$ = 26.4 GPa; $E_2$ = 7.17 GPa; $G_{12}$ = 5.12 GPa; $\nu_{12}$ = .267) with aligned cracks.